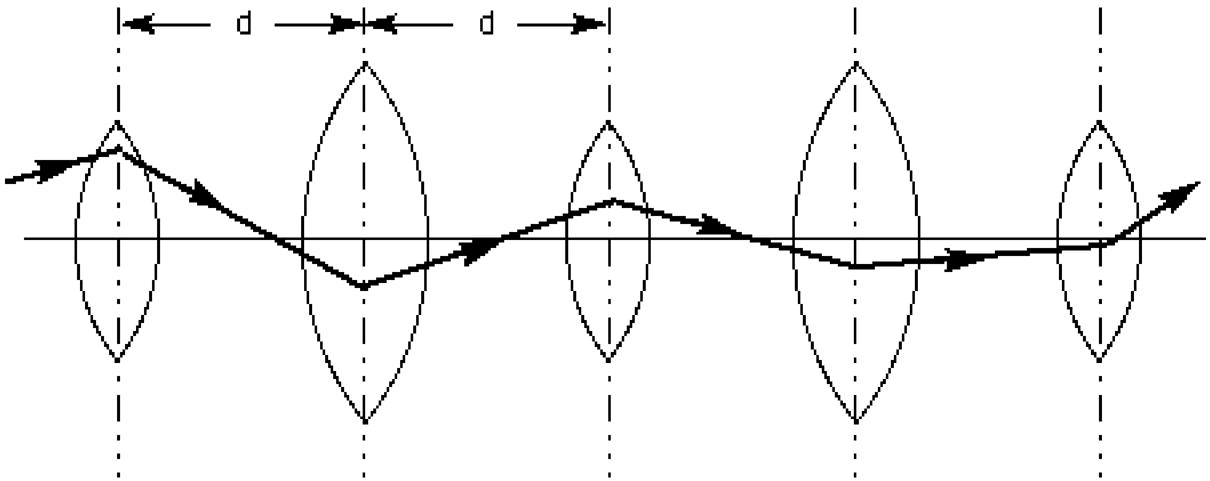
IV. OPTICAL RESONATORS:

STABILITY CRITERIA FOR RESONATORS AND PERIODIC OPTICAL STRUCTURES BY RAY OPTIC ANALYSIS

Consider a prototypical periodic guiding lens system or an equivalent resonator.



Using the appropriate **ABCD** matrix with the indicated reference planes [17], we may write

$$\rho_{m+1} = A\,\rho_m + B\,\rho'_m \qquad \text{[IV-1a]}$$

and

$$\rho'_{m+1} = C\,\rho_m + D\,\rho'_m \qquad \text{[IV-1b]}$$

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[17] Where for reference , we see that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} = \begin{bmatrix} 1-d/f_2-2\,d/f_1+d^2/f_1f_2 & 2d-d^2/f_2 \\ d/f_1f_2-1/f_1-1/f_2 & 1-d/f_2 \end{bmatrix}$$

From the first equation we write

$$\rho'_m = \frac{\rho_{m+1} - A\,\rho_m}{B} \quad \text{and} \quad \rho'_{m+1} = \frac{\rho_{m+2} - A\,\rho_{m+1}}{B} \,.$$

and substitute into Equation [IV-1b] to obtain

$$\rho_{m+2} - [A + D]\,\rho_{m+1} + [AD - BC]\,\rho_{m+1} = 0 \qquad \text{[IV-2a]}$$

The determinant of the coefficients $|A\,D - B\,C| = 1$ so that

$$\rho_{m+2} - [A + D]\,\rho_{m+1} + \rho_{m+1} = 0 \;. \qquad \text{[IV-2a]}$$

We see that

$$\frac{A+D}{2} = \beta = 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2\,f_1\,f_2} = -1 + 2\left(1 - \frac{d}{2\,f_1}\right)\left(1 - \frac{d}{2\,f_2}\right) \qquad \text{[IV-3]}$$

Thus, stable ray propagation may characterized by **bound solutions** of the form $\rho_m = \rho_0\,\exp(i\,m\varphi)$ which are possible if and only if

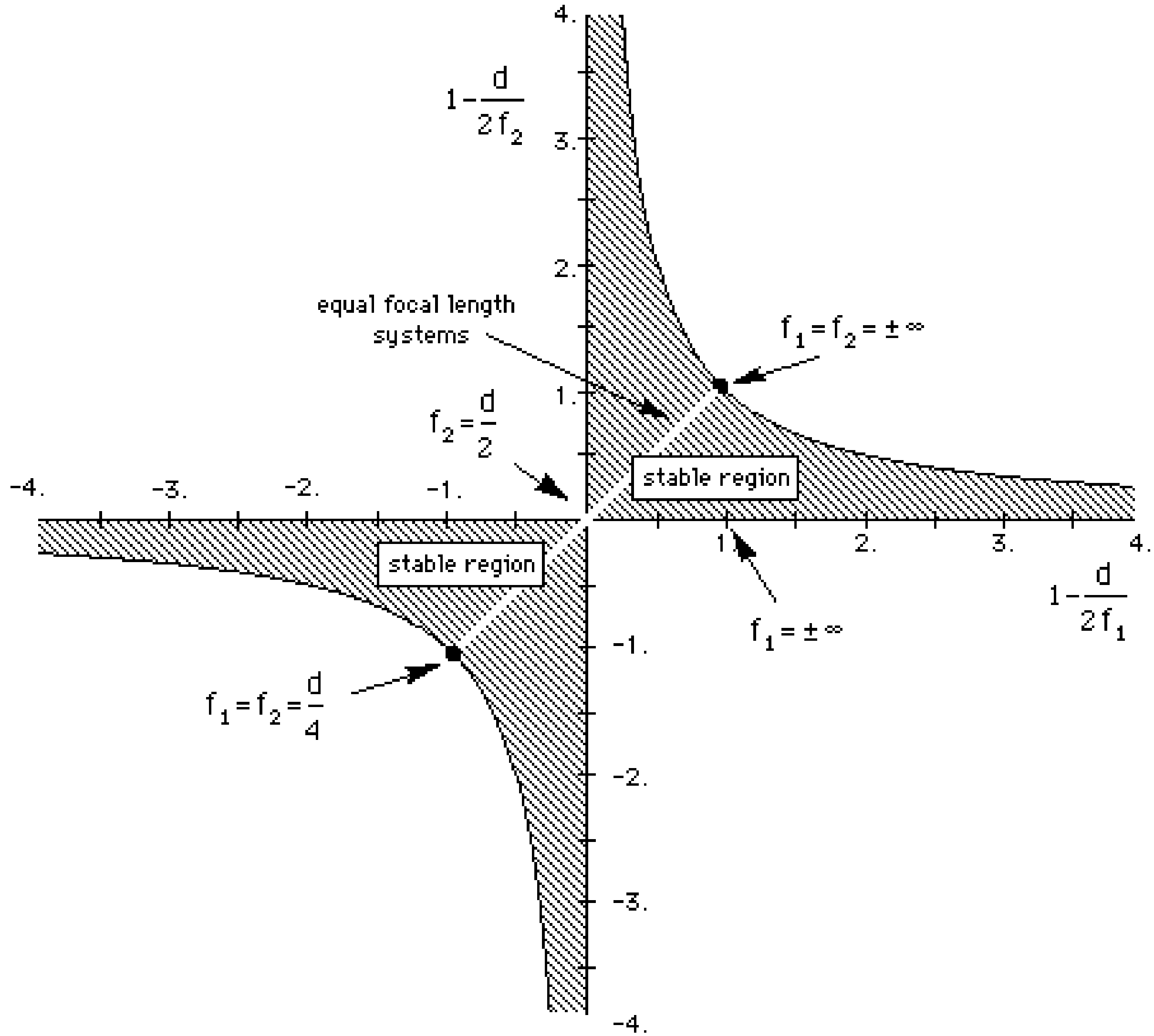
$$\exp(i\varphi) + \exp(-i\,\varphi) = 2\,\cos\varphi = A + D = 2\beta \quad \text{or} \quad \cos\varphi = \beta \qquad \text{[IV-4]}$$

Therefore propagation is stable -- *i.e.* the rays are confined -- when $|\,\beta\,| \le 1$ so that

$$0 \le \left(1 - \frac{d}{2\,f_1}\right)\left(1 - \frac{d}{2\,f_2}\right) \le 1 \;. \qquad \text{[IV-5]}$$

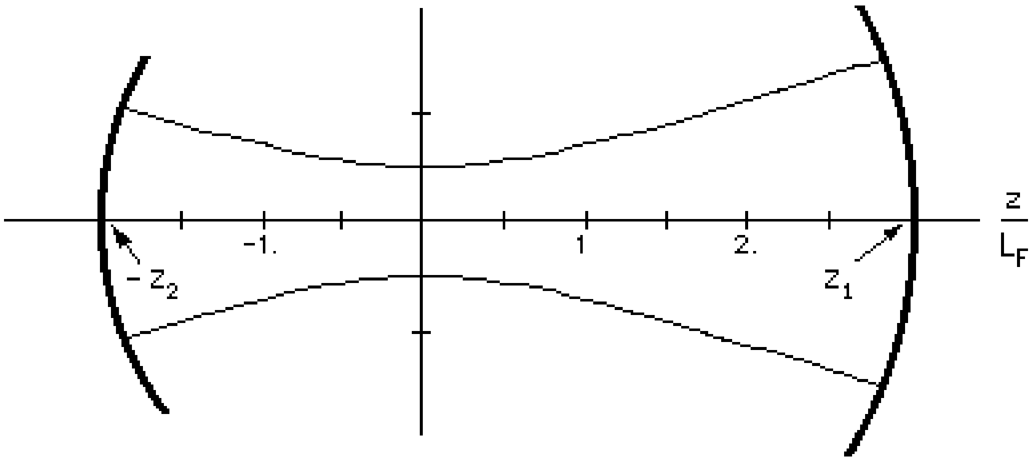
Ray stability of rays in a periodic system may be usefully characterized in terms of the variables $u_1 = 1 - \dfrac{d}{2\,f_1}$ and $u_2 = 1 - \dfrac{d}{2\,f_2}$ as follows:

Stability (Confinement) Diagram for Periodic Systems



STABILITY OF A SPHERICAL MIRROR RESONATORS -- USING SOLUTIONS OF THE PARAXIAL WAVE EQUATION:

Consider a Hermite-Gaussian mode confined in an **asymmetrical spherical cavity**:



In order to sustain a resonant mode in such a cavity, the beam's radius of curvature must match each mirror's radius of curvature at the mirror's surface and, thus, the following conditions must hold (see Equation [III-20]):

$$R_1 = z_1 + \frac{L_F^2}{z_1} \quad \text{and} \quad R_2 = -z_2 - \frac{L_F^2}{z_2} \qquad \text{[IV-5a]}$$

where $d = z_1 + z_2$. Hence, we see that

$$z_1 = \frac{R_1 \pm \sqrt{R_1^2 - 4L_F^2}}{2} \quad \text{and} \quad z_2 = \frac{-R_2 \pm \sqrt{R_2^2 - 4L_F^2}}{2} \;. \qquad \text{[IV-5b]}$$

with a lot of algebra we can show that

$$L_F^2 = \left[\frac{\pi w^2(0)}{\lambda}\right]^2 = \frac{-d u_1 u_2 R_1 R_2 \left[d + u_1 R_1 - u_2 R_2\right]}{\left[u_1 R_1 - u_2 R_2\right]^2} \qquad \text{[IV-6]}$$

where now $u_1 = 1 - d/R_1$ and $u_2 = 1 + d/R_2$.

For a **symmetric resonator** $R_2 = -R_1$ and $u_1 = u_2$

$$L_F^2 = \left[\frac{\pi w^2(0)}{\lambda}\right]^2 = \frac{d\left[2R - d\right]}{4} \qquad \text{[IV-7a]}$$

and

$$w(z_1) = w(-z_2) = \left[\frac{d\lambda}{2\pi}\right]^{1/2} \left[\frac{2R^2}{d(R - d/2)}\right]^{1/4} \qquad \text{[IV-7b]}$$

For an **asymmetric resonator**, it can be shown with a bit more algebra that

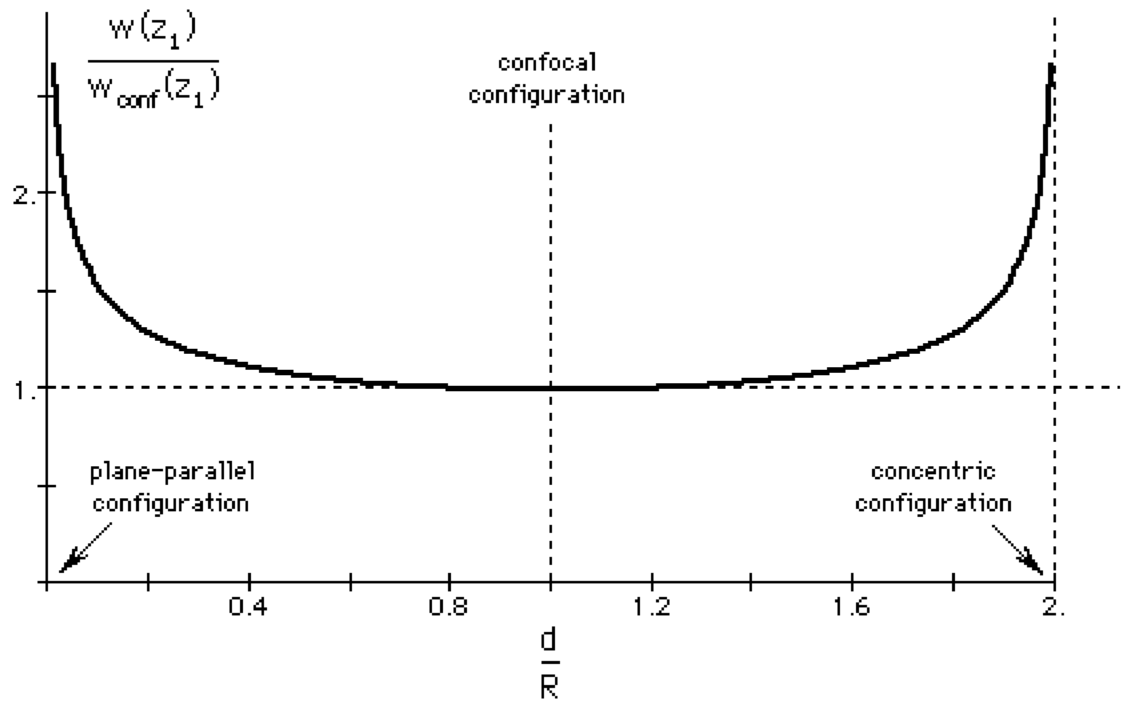
$$w(z_1) = \left[\frac{d\lambda}{\pi}\right]^{1/2} \left[\frac{u_2}{u_1(1 - u_1 u_2)}\right]^{1/4} \qquad \text{[IV-8a]}$$

$$w(-z_2) = \left[\frac{d\lambda}{\pi}\right]^{1/2} \left[\frac{u_1}{u_2(1 - u_1 u_2)}\right]^{1/4} \qquad \text{[IV-8b]}$$

As a measure of the effect of resonator length and mirror radius on **diffraction loss** consider the ratio:

$$\frac{w(z_1)}{w_{\text{conf}}(z_1)} = \frac{w(-z_2)}{w_{\text{conf}}(-z_2)} = \left[\frac{d}{R}\left(1 - \frac{d}{2R}\right)\right]^{-1/4} \qquad \text{[IV-9]}$$

where $w_{\text{conf}}(z_1) = \sqrt{2}\; w(0)$ is beam width at the mirror for the confocal configuration -- *i.e.*, when both mirrors have their focal points at the mid-point of the cavity



Resonance Frequencies of the Optical Resonator:

From Equation [III-28], we see the the “round-trip” cold resonance condition for a Hermite-Gaussian mode is given by

$$k\,d - (n + m + 1)\left[\tan^{-1}(z_1/L_F) + \tan^{-1}(z_2/L_F)\right] = N\pi \qquad \text{[IV-10a]}$$

where N is an integer. In terms of frequency, the resonce condition is

$$\omega = \frac{c}{d}\left\{N\pi + (n+m+1)\left[\tan^{-1}(z_1/L_F) + \tan^{-1}(z_2/L_F)\right]\right\} \qquad \text{[IV-10b]}$$

After much algebra, it can be shown that:

$$\omega = \frac{c}{d}\left[N\pi + (n+m+1)\cos\sqrt{u_1 u_2}\right] \qquad \text{[IV-10b]}$$